Int. J. Multiphase Flow Vol. 19, No. 2, pp. 329–336, 1993
Printed in Great Britain.

0301-9322/93 \$6.00 + 0.00

THEORETICAL PREDICTION OF THE CRITICAL WEBER NUMBER

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(Received 5 January 1991; in revised form 15 October 1992)

Abstract—The initial stages of the bag-mode liquid droplet breakup in a fluid flow are studied in both a liquid–gas system, e.g. a water or molten metal droplet in air, and a liquid–liquid system, e.g. an oil or mercury droplet in water. The Weber number and its critical value play a fundamental role in this consideration. An individual droplet in uniform flow is investigated. The droplet viscosity and flow compressibility are taken into account. The critical Weber number is shown to be dependent upon the Ohnesorge number and the fluid/droplet density ratio for both an incompressible fluid and supersonic flow.

Key Words: critical Weber number, instantaneous Weber number, bag-mode droplet breakup, rim/original droplet mass ratio

1. INTRODUCTION

There is a wealth of literature on droplet breakup but they are mainly experimental works. The author is not aware of any contemporary theoretical publication. There are known to be several modes of droplet breakup. This paper considers one of them, the so-called bag-mode which occurs at small Weber numbers (We). In this mode the central part of a flattened droplet is blown up downstream into a bag, thus the droplet constitutes a heavy toroidal rim with a thin bag. The rim and bag expand; first, the rim breaks, producing a great number of fine droplets, then the bag disintegrates, producing larger droplets. The very complex process of droplet breakup is over. Let n be the rim/original droplet mass ratio. According to Lane (1951) and Lefebvre (1989), $n = 0.7$; according to Hanson *et al.* (1983), $n = 0.75$.

What about the value of the critical Weber number We_*, below which the droplet does not break up? Pilch & Erdman (1987) cited the empirical relation

$$We_* = 12(1 + 1.077\,On^{1.6}), \quad [1]$$

where On is the Ohnesorge number, and stated that when $On < 0.1$, the droplet viscosity may be neglected and $We_* = 12$. Hanson *et al.* (1963) reported that the condition $10 < We_* < 16$ is fulfilled and the largest value corresponds to the smallest original droplet diameter and the highest droplet viscosity.

2. SUBJECT OF THE PAPER

The appropriate assumptions and definitions must be presented first. Assume that, at the instant $t = 0$, a droplet of original diameter D and velocity $u = 0$ is introduced into a horizontal uniform liquid or gas flow of velocity V and density ρ. The Weber number is determined as

$$We = \rho\frac{V^2D}{\sigma}, \quad [2]$$

where σ is the droplet surface tension with respect to the fluid in the flow. The droplet is accelerated, so one needs to define the instantaneous Weber number:

$$We_i = (1 - \bar{u})^2 We, \quad [3]$$

where $\bar{u} = u/V$.

Figure 1 shows the family of curves

$$We_i = f(\bar{d}, We), \quad [4]$$

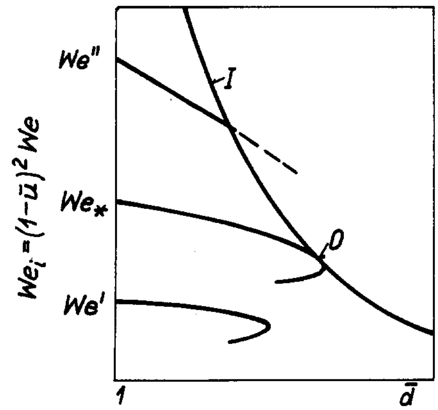

Figure 1. Accelerating and flattening of a droplet: I—the beginning of the blowing up of the bag; 0—radial velocity equal to zero.

where $\bar{d} = d/D$; d is the instantaneous droplet diameter and We is the family parameter. It is seen that, at a sufficiently low Weber number We′, a droplet is reversibly flattened; i.e. it returns to its original spherical shape. At a sufficiently large Weber number We″, a droplet will break up. Curve I represents the beginning of breakup, i.e. the initial blowing up of the bag. At present, it is possible to give a precise definition of the critical Weber number, namely that it is the value We_* of the family parameter We when the curve $We_i = f(\bar{d}, We_*)$ is tangent to curve I. It is seen that at the point of tangency 0, the radial velocity inside the droplet is zero.

The Ohnesorge number is expressed as

$$On = \frac{\mu_d}{\sqrt{\rho_d D \sigma}}, \qquad [5]$$

where μ_d and ρ_d are the viscosity and density of the droplet, respectively. For instance, at a temperature of 20°C and $D = 0.1$ mm: for water, On = 0.012; and for 100 cstk oil, On = 2.5.

The fluid/droplet density ratio $\epsilon = \rho/\rho_d$ is also used herein, some example values are listed in table 1.

In this work, first assumptions are made about the shape of a flattened dropiet and then a radial motion inside the droplet is studied. The relationship

$$We_i = F(\bar{d}, We, On, \epsilon, Ma), \qquad [6]$$

where Ma is the Mach number, is derived. The extreme cases Ma = 0 and Ma > are considered. From [6] it is concluded that We_* is a function of three parameters: On, ϵ, Ma. Note that curve I in figure 1 depends solely upon Ma.

3. MOTION INSIDE A DROPLET

3.1. The shape of a droplet

It is assumed that a flattened droplet takes the shape of a flat disk with rounded edges—see figure 2. From Guldin's theorem we obtain the relation between the radii of the disk

$$\bar{R} = \left(F - \frac{\pi}{4}\right)\bar{r}, \qquad [7]$$

Table 1

ϵ	Example
0.0001	Molten metal in air
0.001	Water in air
0.01	Diesel oil in compressed air
0.1	Mercury in water
1	Water in oil

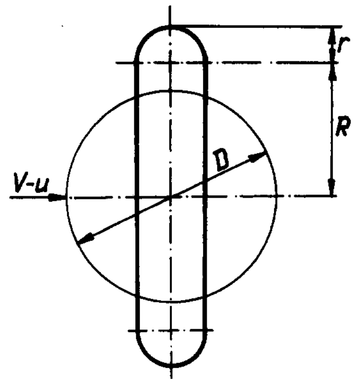

Figure 2. Assumed shape of a flattened droplet.

where $\bar{R} = R/D$, $\bar{r} = r/D$ and

$$F = \left(\frac{1}{12\bar{r}^3} - \frac{2}{3} + \frac{\pi^2}{16}\right)^{0.5}. \qquad [8]$$

3.2. Motion of the rounded part of a disk

The disk consists of two parts: a flat part and a rounded one. The radial motion of the rounded part will be studied here. It is postulated that this part of varying mass moves radially with velocity

$$w = \frac{dR}{dt}. \qquad [9]$$

By means of figure 3, one may formulate the momentum equation

$$\frac{d}{dt}(w\,dm) = P_1 - P_2\,d\varphi - 2P_3 + P_4 + P_5\,d\varphi, \qquad [10]$$

where

$$dm = \frac{\rho_d}{2\pi}\left(\frac{\pi D^3}{6} - 2\pi r R^2\right)d\varphi$$

and the forces will be determined successively.

The aerodynamic force is expressed as

$$P_1 = lq2r\left(R + \frac{r}{2}\right)d\varphi,$$

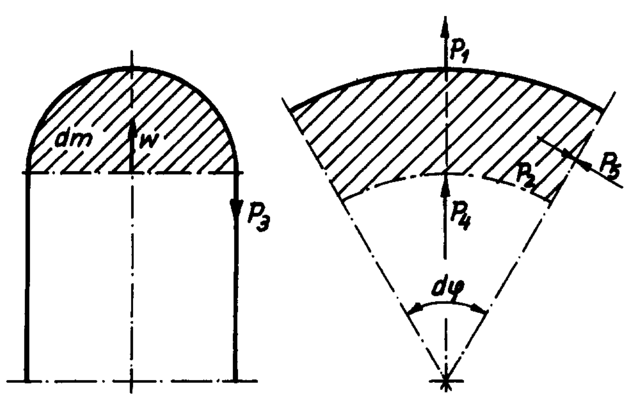

Figure 3. Forces in the rounded part of a disk.

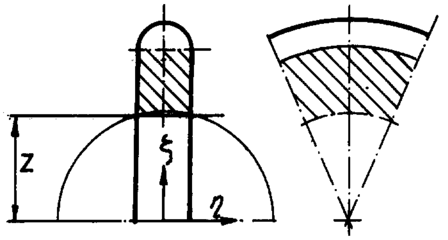

Figure 4. Viscous flow in the flat part of a disk—determination of the "duct".

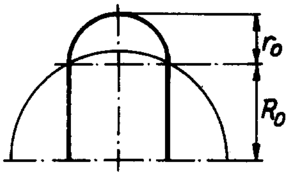

Figure 5. A "duct" of zero length.

where l is the aerodynamic coefficient due to a pressure distribution on the surface of the rounded part of the disk and $q = \rho(V - u)^2/2$ is an instantaneous dynamic pressure. There are two surface tension forces:

$$P_2 = \sigma\pi r \quad \text{and} \quad P_3 = \sigma R \, d\varphi.$$

The residual forces, i.e. pressure forces P_4 and P_5, are determined after taking into consideration a viscous flow inside the flat part of the disk.

Using figures 4 and 5, we shall investigate a radial laminar flow in a "duct" between the radii

$$\bar{z} = (0.25 - \bar{r})^{0.5} \qquad [11]$$

and $R \geqslant R_0$, where $\bar{R}_0 = 0.452$ and $\bar{r}_0 = 0.215$ are calculated from [7] by trial-and-error.

Using figure 6, we formulate the momentum equation of the shaded element

$$(r - \eta)\, dp = \tau \, d\zeta,$$

where

$$\tau = \mu_d \frac{dv}{d\eta}$$

and the boundary condition has the form $\eta = v = 0$. We obtain the velocity distribution

$$v = \frac{1}{\mu_d}\left(r\eta - \frac{\eta^2}{2}\right)\frac{dp}{d\zeta}$$

and the volume flow

$$dQ = \frac{2}{3}\zeta\frac{r^3}{\mu_d}\frac{dp}{d\zeta}\, d\varphi.$$

On the other hand, the volume flow is equal to $2wrR\, d\varphi$, where w is determined by [9], hence we find the relation

$$dp = \frac{3wR\mu_d}{r^2}\frac{d\zeta}{\zeta},$$

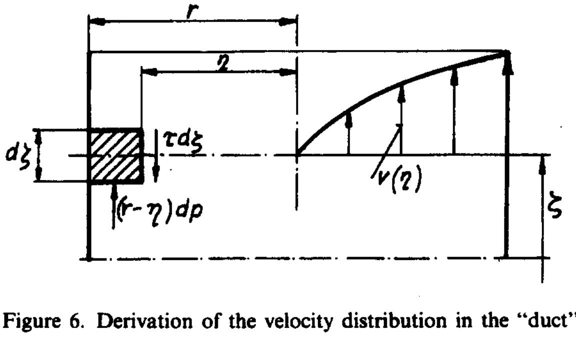

Figure 6. Derivation of the velocity distribution in the "duct".

which integrated from z to R takes the desired form

$$\Delta p = \frac{3wR\mu_d}{r^2}\ln\frac{R}{z}, \qquad [12]$$

where z is defined by [11].

At last, it is possible to express the pressure forces P_4 and P_5. We return to figure 6 and assume that at $\zeta = z$ the pressure is equal to the dynamic pressure q and at $\zeta = R$ the pressure is equal to $(q - \Delta p)$, where Δp is expressed by [12]. Therefore, we find

$$P_4 = (q - \Delta p)2rR\,d\varphi.$$

When $\zeta = R + r$, then the pressure inside the droplet is defined by the Laplace formula

$$p_\sigma = \sigma\left(\frac{1}{R} + \frac{1}{R+r}\right). \qquad [13]$$

Now, we make the simplifying assumption: in the rounded part of the disk there is a constant pressure equal to the arithmetic mean of the pressure values at $\zeta = R$ and $\zeta = R + r$, hence we can write

$$P_5 = \tfrac{1}{2}(q - \Delta p + p_\sigma)\frac{\pi r^2}{2},$$

when p_σ is defined by [13].

4. GOVERNING EQUATIONS

We rewrite [9] in the dimensionless form

$$\frac{d\bar{r}}{dT} = \frac{\bar{w}}{\sqrt{\epsilon}}\frac{d\bar{r}}{d\bar{R}}, \qquad [14]$$

where

$$\bar{w} = w/V,$$

$$T = \frac{V}{D}\sqrt{\epsilon}t,$$

$$\frac{d\bar{r}}{d\bar{R}} = -F\left(\frac{1}{24\bar{r}^3} + \frac{\pi}{4}F + \frac{2}{3} - \frac{\pi^2}{16}\right)^{-1}$$

and F is defined by [8].

Similarly, we derive

$$\left(\frac{1}{12} - \bar{r}\bar{R}^2\right)\frac{d\bar{w}}{dT} - \frac{\bar{w}^2\bar{R}^2}{\sqrt{\epsilon}}\left(2\frac{\bar{r}}{\bar{R}} + \frac{d\bar{r}}{d\bar{R}}\right) = \sqrt{\epsilon}(1-\bar{u})^2\bar{r}^2\left[(1+l)\frac{\bar{R}}{\bar{r}} + \frac{l}{2} + \frac{\pi}{8}\right] +$$

$$- \frac{\delta \mathrm{On}}{\sqrt{\mathrm{We}}}\bar{w}\bar{R}\ln\frac{\bar{R}}{\sqrt{0.25 - \bar{r}^2}}\left(6\frac{\bar{R}}{\bar{r}} + \frac{3}{4}\pi\right) + \frac{\pi}{4}\frac{\sqrt{\epsilon}}{\mathrm{We}}\bar{r}\left(\frac{\bar{r}}{\bar{r}+\bar{R}} - \frac{8}{\pi}\frac{\bar{R}}{\bar{r}} - 3\right), \qquad [15]$$

where $\delta = 0$ if $\bar{r} \geqslant 0.215$ and $\delta = 1$ lf $\bar{r} < 0.215$; We and On are determined by [2] and [5], respectively; $\bar{r}$ and $\bar{R}$ are connected by [7].

We formulate a translatory momentum equation which has the dimensionless form

$$\frac{d\bar{u}}{dT} = 3\sqrt{\epsilon}[C_{d1}\bar{R}^2 + C_{d2}(2\bar{r}\bar{R} + \bar{r}^2)](1-\bar{u})^2, \qquad [16]$$

Table 2

Flow	C_{d1}	C_{d2}
Ma = 0	1	0.4
Ma > 1	1.4	1

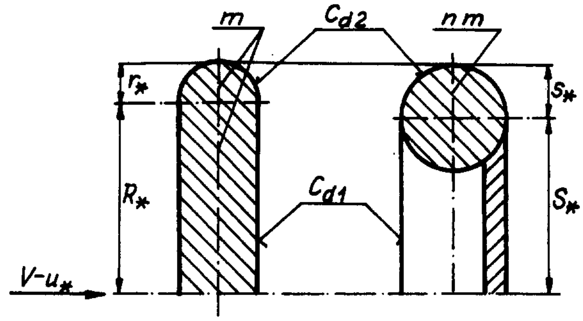

Figure 7. Transition at We_* of the disk into the torus with a flat bag.

where C_{d1} and C_{d2} are the drag coefficients of the flat and rounded parts, respectively, of the disk. According to Hetsroni (1982), one may assume values of the drag coefficients, as in table 2.

Relations [14]–[16] constitute the equations governing the flattening of a droplet. There are: an argument T; three unknowns $\bar{r}, \bar{u}, \bar{w}$; three parameters We, On, ϵ; and three material constants l, C_{d1}, C_{d2} which depend upon Ma. The above equations have the initial condition:

$$T = \bar{u} = \bar{w} = 0, \quad \bar{r} = 0.5$$

and may be integrated to curve I in figure 1.

5. THE BEGINNING OF THE BLOWING UP OF THE BAG

Returning to figure 1, we repeat: at the point of tangency 0, the radial velocity becomes zero and a bag begins to appear (blow up) downstream. Our aim is solely to determine curve I. To this end, we postulate that, at the point 0, a disk with rounded edges travels into the torus with a flat

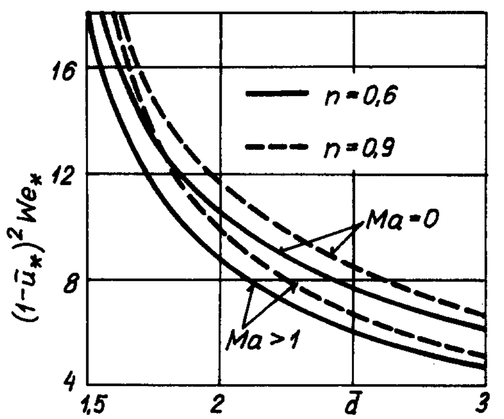

Figure 8. Beginning of the blowing up of the bag at We_*.

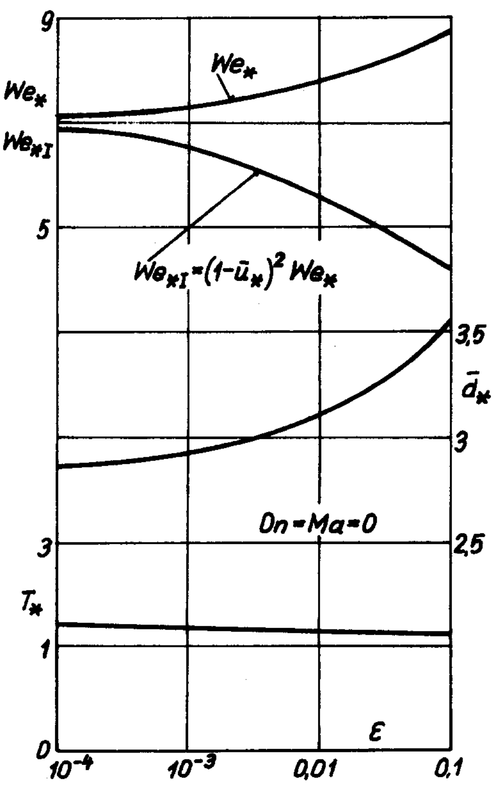

Figure 9. Influence of ϵ on droplet breakup.

bag, as figure 7 shows, it has a constant diameter and a given value of the ratio n. Therefore, we can write the relations

$$\bar{R}_* + \bar{r}_* = \bar{S}_* + \bar{s}_*; \quad \bar{S}_* \bar{s}_*^2 = n/12\pi. \qquad [17]$$

Naturally, during the transition the total energy is unchanged: the kinetic energy of the radial motion is simply equal to zero and the surface areas of the disk and torus are nearly the same.

On the torus with the flat bag a drag is exerted. The bag is held by the surface tension. When the bag begins to blow up, then the following condition must be fulfilled:

$$C_{d1} q_* \pi S_*^2 - C_{d2} q_* \pi [(S_* + s_*)^2 - S_*^2] = 4\pi\sigma S_*,$$

which may be rewritten as

$$C_{d1} \bar{S}_* - C_{d2}\left(2\bar{s}_* + \frac{\bar{s}_*^2}{\bar{S}_*}\right) = \frac{8}{(1-\bar{u}_*)^2 We_*}. \qquad [18]$$

By means of the above condition and [17] the curves in figure 8 have been plotted, they are clearly curve I from figure 1. As has been demonstrated, curve I depends upon the flow compressibility and the ratio n. Note that in the experiment a value of $n = 0.7$ to 0.75 was observed.

6. RESULTS

As has been shown, We_* is a function of the constants l, n, C_{d1} and C_{d2} and of the parameters ϵ, On and Ma. The values $l = 0.59$ and $n = 0.75$ are fixed and the values of C_{d1} and C_{d2} are taken from table 2. Let We_{*l}, $\bar{d}_*$ and T_* denote the instantaneous Weber number and dimensionless droplet diameter and time corresponding to the beginning of the blowing up of the bag at We_*.

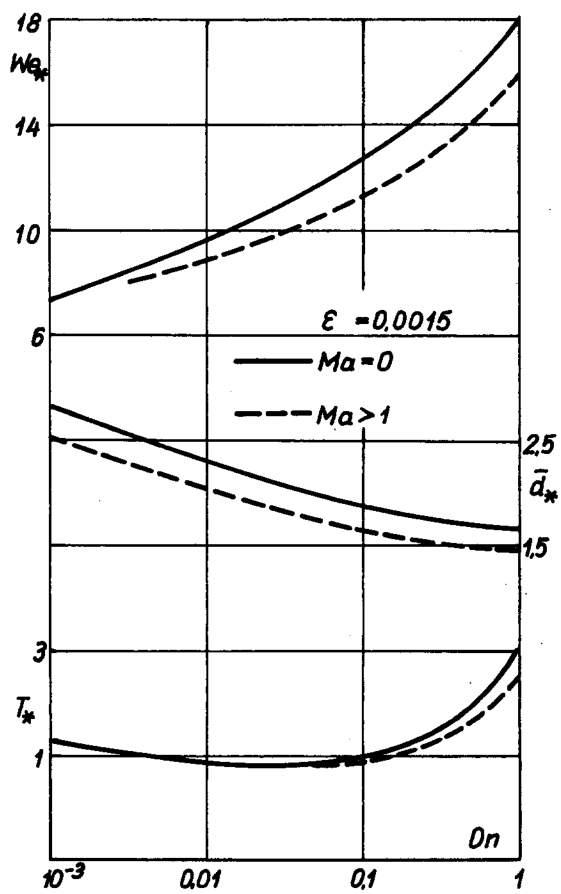

Figure 10. Influence of On and Ma on droplet breakup.

The above quantities are plotted in figure 9 as functions of ϵ and in figure 10 as function of On. Figure 10 also shows the influence of the flow compressibility.

7. CONCLUSIONS

1. The influence of ϵ and Ma on We_* is negligible.
2. The influence of On on We_* is significant over the whole range of On; it is not true that when On < 0.1 then $\mathrm{We}_* = 12$.
3. The time T_* depends merely upon On.
4. This paper deals with an individual droplet in uniform flow. In the opposite case, there is a mutual interaction between the droplets and flow (e.g. Tarnogrodzki & Pyzik 1991) and the above conclusions need to be modified or changed.

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